Last class: wave equation for circular membrane R

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

boundary cond. $u|_{\partial R} = 0$

$$|u(\underbrace{0,0}_{r=0 \text{ in polar coordinates}},t)| < \infty \quad \forall t$$

used polar coordinates:

$$u(r,\theta,t) = f(r)\, g(\theta)\, h(t)$$

we got 3 ODE's

① $h''(t) = -\lambda c^2 h(t)$

② $g''(\theta) = -\mu g(\theta)$

with bd. conditions $g(\pi) = g(-\pi)$, $g'(\pi) = g'(-\pi)$

π and $-\pi$ describe same angle

$\Rightarrow \boxed{\mu = m^2, \quad m = 1, 2 \ldots}$ as for Laplace's equation for disk

③ $r \frac{d}{dr}\left(r \frac{df}{dr}\right) + (\lambda r^2 - m^2) f = 0$ $\quad \boxed{|f(0)| < \infty}$

$f' = \frac{df}{dr}$

can be rewritten as $\boxed{r^2 f'' + r f' + (\lambda r^2 - m^2) f = 0}$

③ can be transformed into Bessel's ODE via substitution $z = \sqrt{\lambda}\, r$

$\Rightarrow \boxed{\frac{df}{dr} = \frac{df}{dz}\frac{dz}{dr} = \frac{df}{dz} \cdot \sqrt{\lambda}}$ $\quad \boxed{\frac{d^2f}{dr^2} = \lambda \frac{d^2f}{dz^2}}$

substitute into yellow equ.

$\Rightarrow \underbrace{r^2\lambda}_{=z^2} \frac{d^2f}{dz^2} + \underbrace{r\sqrt{\lambda}}_{=z} \frac{df}{dz} + (\underbrace{\lambda r^2}_{=z^2} - m^2) f = 0 \Rightarrow \boxed{z^2 \frac{d^2f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0}$

need to study Bessel's ODE (last equ. on previous page)

solutions can not be described by ordinary functions

can be described by power series.

First step: consider simplified problem

$$z^2 f'' + z f' - m^2 f = 0$$

has solutions of the form $f(z) = z^s$

plug into ODE $\Rightarrow$

get equation $(s(s-1) + s - m^2)\, z^s = 0$

$\Rightarrow$ $s^2 - m^2 = 0$

$\Rightarrow$ $\boxed{s = \pm m}$

Can deduce from this:

There exist two linearly independent solutions of Bessel's equation of the form

$$J_m(z) = \sum_{j=0}^{\infty} a_j z^{j+m} \qquad m>0$$

$$Y_m(z) = \sum_{j=0}^{\infty} b_j z^{j-m}$$

a_j's and b_j's can be calculated via recursive relations.

Similar functions for $m=0$

<u>Observe:</u> $\lim_{z\to 0} |J_m(z)| = 0$ for $m>0$

$\lim_{z\to 0} |Y_m(z)| = \infty$

our cond. $f(0)$ bounded

$\Rightarrow$ implies: we can only use the function $J_m(z)$.

$J_m(z)$ = Bessel functions of first kind.

Result: $f(r) = J_m(\sqrt{\lambda}\, r)$

for some m.

Recall: we considered solutions of the form

$u(r,\theta,t) = f(r)\, g(\theta)\, h(t)$

BC $u(a,\theta,t) = 0 \Rightarrow \boxed{f(a) = 0}$

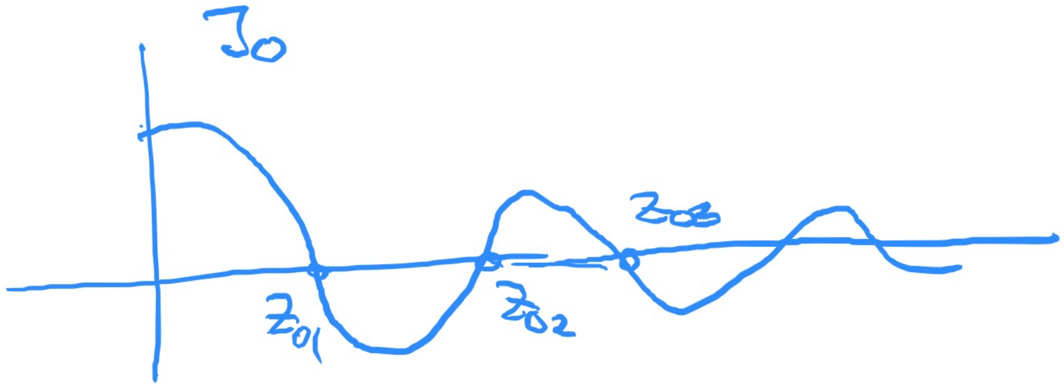
$\Rightarrow \boxed{f(a) = J_m(\sqrt{\lambda}\, a) = 0}$

Need to know zeros of Bessel function

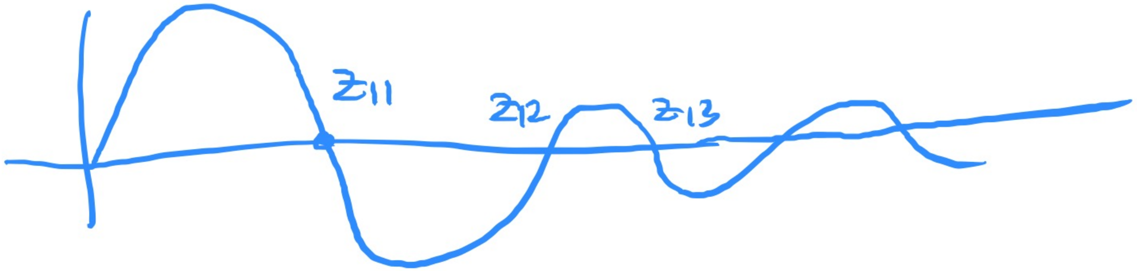
fact: J_m has infinitely many zeros

$\lambda_{m,1}, \lambda_{m,2}, \lambda_{m,3}, \dots$

e.g.



J_1



$$0 = f(a) = J_m(\sqrt{\lambda}\, a)$$

$$\Rightarrow \sqrt{\lambda}\, a = z_{mn} \quad \text{for some } n$$

$$\Rightarrow \boxed{\lambda = \frac{z_{mn}^2}{a^2}}$$

we get the following product solutions

$$u(r,\theta,t) = f(r)\, g(\theta)\, h(t)$$

$$\boxed{g(\theta) = a_m \cos m\theta + b_m \sin m\theta}$$

$$\Rightarrow \boxed{f(r) = J_m(\sqrt{\lambda_{mn}}\, r)} \qquad \sqrt{\lambda_{mn}} = \frac{z_{mn}}{a}$$

z_{mn} = n-th zero of J_m

$$\boxed{h(t) = a_{mn} \cos \sqrt{\lambda_{mn}}\, ct + b_{mn} \sin \sqrt{\lambda_{mn}}\, ct}$$

General power series solution

$$\boxed{u(r,\theta,t) = \sum_{m,n} J_m(\sqrt{\lambda_{mn}}\, r)\left(a_m \cos m\theta + b_m \sin m\theta\right)\left(a_{mn} \cos \sqrt{\lambda_{mn}}\, ct + b_{mn} \sin \sqrt{\lambda_{mn}}\, ct\right)}$$

how to express Bessel functions as power series?

set up the solution of ODE as a power series

subtle point: Bessel's equation is of the form

$$z^2 f'' + z f' + (z^2 - m^2) f = 0$$

$z \neq 0 \implies$ $$f'' + \frac{1}{z} f' + \left(1 - \frac{m^2}{z^2}\right) f = 0$$

$z = 0$ is problematic point.

we will see next time:

if we set up power series as $\boxed{\sum a_j z^{m+j} = f(z)}$

and $\sum b_j z^{j-m}$

and plug into Bessel's eqn.

we can calculate coeff. a_j's and b_j's via recursive relation get $\quad f'(z) = \sum (m+j) a_j z^{j+m-1}$

$$f''(z) = \sum (m+j)(m+j-1) a_j z^{j+m-2}$$